Cubical Models Are Cofreely Parametric

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Foundations for mathematics

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Type theory [Martin-Löf 1972]

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$(n : \mathbb{N}) \to (p : \mathbb{N}) \times (p \text{ } \mathit{is\ prime}) \times (p > n)$	_

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Type theory has more diverse models than set theory.

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Two relevant flavors:

Computational models [Curry-Howard 1969]

Terms are interpreted as computer programs.

Homotopical models [Awodey, Warren 2009]

Types are interpreted as spaces, with equality interpreted as paths.

Easy to build new models from old ones (presheaf, slice, gluing...).

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Assume given a model $\mathcal{C}$.

Direct application

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The model $\mathcal{C}$ also interprets unprovable principles.

Reverse application

Such principles can be safely added to type theory.

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Consequence

In a homotopical model, equivalent types are equal.

This is called the univalence axiom.

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Some computational models enjoy a principle called parametricity.

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Consequence [Reynolds 83]

Some computational models enjoy a principle called parametricity.

Definition

A model of type theory is called parametric if:

- Any type comes with a relation.
- Any term respects these relations.

Cubical structure

A semi-cubical structure on a type X consists of:

- For any $x, y : X$, a type of path between them.
- For any four paths drawing a square, a type of fillers for this square.
- And so on.

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This structure originates from a homotopical context [Kan 1955].

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- A presheaf model of parametric type theory.
  [Bernardy, Coquand, Moulin 2015]
- Cubical categories for higher-dimensional parametricity.
  [Johann, Sojakova 2017]
- Internal parametricity for cubical type theory.
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Question

How can we explain this phenomenon?

Dealing with many variants of cubes is part of the challenge.

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- But this relation is itself a type, so it comes with a relation.
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Basic insight

This iteration gives a semi-cubical structure.

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By analyzing this basic insight, we see a correspondence:

Variants of parametricity $\Leftrightarrow$ Homotopical structures

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We develop a theory for variants of parametricity, such that:

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Given a model $\mathcal{C}$, there is a 'largest' parametric model in $\mathcal{C}$.

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Main thesis

Cubical models are cofreely parametric.

Contributions

We present two frameworks:

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1. Parametricity as an extension by section

- An extension by section adds inductively-defined unary operations to a theory.
- The functor forgetting these operations has a right adjoint.
- Examples of extensions by section:
  - Parametricity for clans.
  - Parametricity for categories with families.
  - Parametricity for categories with families with arrow types and a universe.

2. Parametricity as a module structure

- Use a symmetric monoidal closed category of models.
- Define parametric models as modules.
- Describe cofreely parametric models as cofree modules.
- Examples of cofree modules:
  - Categories of cubical objects, for any kind of cubes.
  - Lex categories of truncated cubical objects.
  - Clans of Reedy fibrant cubical objects.

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Both frameworks cover many examples unrelated to parametricity.

Part 1:

Parametricity as an Extension by Section

The origins of parametricity

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The origins of parametricity

By induction on types and terms:

- System F is parametric [Reynolds 83].
- Various type theories are parametric [Bernardy, Jansson, Paterson 2010], [Keller, Lasson 2012], $\cdots$

Parametricity and semi-cubical types [Moeneclaey 2021]

- Axiomatized parametricity as inductively-defined.
- Proved that cofreely parametric models exist.

In this part we give an alternative presentation.

Plan for Part 1

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- Parametricity is an extension by section of categories with families (with arrow types and a universe).

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Conclusion

Cofreely parametric categories with families (with arrow types and a universe) exist.

Inductive definitions

We introduce signatures for quotient inductive-inductive types [Kaposi, Kovács, Altenkirch 2019], [Kovács, Kaposi 2020].

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Signatures are contexts in a type theory with:

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- A universe $\mathcal{U}$ closed under them.
- Arrow types with domain in $\mathcal{U}$.

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Example: Signature for semi-groups

$$
\begin{aligned}
A &: \mathcal{U} \\
m &: A \to A \to A \\
\_ &: (x, y, z : A) \to m(x, m(y, z)) = m(m(x, y), z)
\end{aligned}
$$

We can define inductively on a signature Γ:

- The category Alg_Γ of its algebras.
- The type $Disp_\Gamma(X)$ of displayed algebras over $X : Alg_\Gamma$.
- The type $Sec_\Gamma(X, Y)$ of sections of $Y : Disp_\Gamma(X)$.

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Example with $\Gamma = (A : \mathcal{U})$

Then $X : Alg_{A:\mathcal{U}}$ is simply a type X and we get:

$$\begin{aligned} (Y : X \to \mathcal{U}) &\simeq (X' : \mathcal{U}) \times (p : X' \to X) \\ (x : X) \to Y(x) &\simeq (q : X \to X') \times (p \circ q = id_X) \end{aligned}$$

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Question

Why such a complicated reformulation for initiality?

Example: Natural numbers

The formulation using QIIT is very natural.

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Section	$e : (n : \mathbb{N}) \to P(n)$	$e(0) = 0'$	$e(s(n)) = s'(e(n))$

Extension by section

Let A be a displayed algebra over Γ, internal to signatures.

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This A is an inductive definition, unary and valid for any algebra.

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Definition

The extension of Γ by a section of A is an extension by section.

This extension adds inductively-defined operations.

Categorical extension by section

Definition

A copointed endofunctor on a category $\mathcal{V}$ consists of:

- An endofunctor $E : \mathcal{V} \to \mathcal{V}$.
- A natural transformation $d : E \to Id$.

So any $\mathcal{C} : \mathcal{V}$ comes with $d_{\mathcal{C}} : E(\mathcal{C}) \to \mathcal{C}$.

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Definition

A categorical extension by section is a forgetful functor of the form:

$$CoAlg_{\mathcal{V}}(E, d) \quad \to \quad \mathcal{V}$$

where $\mathcal{V}$ has limits and E commutes with them.

Categorical extension by section from extension by section

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Categorical extension by section from extension by section

Display algebra A over Γ internal to signature	Copointed endofunctor (E, d) of Alg_Γ
Algebra for Γ plus a section of A	$X : Alg_\Gamma$ with a section of d_X.
Functor forgetting the section	$CoAlg(E, d) \to Alg_\Gamma$

Constructing cofree objects

Theorem [folklore, e.g. Kelly 80]

Any categorical extension by section has a right adjoint.

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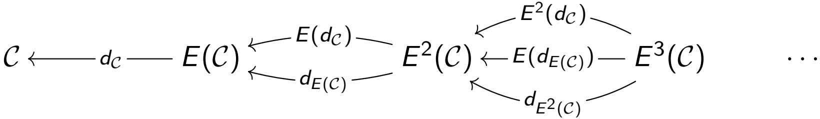
This right adjoint sends $\mathcal{C} : \mathcal{V}$ to the limit of:



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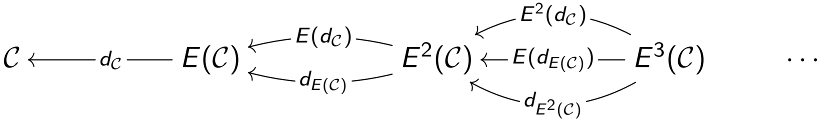

- Gives a right adjoint by the universal property.

Example: Categories

Definition

A parametric category is a category $\mathcal{C}$ equipped with:

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- Morphisms $d^0_\Gamma, d^1_\Gamma : \Gamma_* \to \Gamma$ natural in Γ.

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Proposition

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Proposition

Cofreely parametric categories exist.

Example: Categories with families

A category with families [Dybjer 1995] with product and unit types is called parametric if it is equipped with:

$$\begin{aligned}
_{}_* &: (\Gamma : Ob) \to Ty(\Gamma_0, \Gamma_1) \\
_{}_* &: (\sigma : Hom(\Gamma, \Delta)) \to Tm((\Gamma_0, \Gamma_1, \Gamma_*), \Delta_*[\sigma_0, \sigma_1]) \\
_{}_* &: (A : Ty(\Gamma)) \to Ty(\Gamma_0, \Gamma_1, \Gamma_*, A_0, A_1) \\
_{}_* &: (a : Tm(\Gamma, A)) \to Tm((\Gamma_0, \Gamma_1, \Gamma_*), A_*[a_0, a_1])
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with equations defining $\_{}_*$ inductively on any constructor.

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Cofreely parametric categories with families exist.

Example: Adding arrow types and a universe

Adding arrow types and a universe works fine with parametricity.

For example we can define:

$$\begin{aligned} (A \to B)_*(f_0, f_1) &= (x_0, x_1 : A) \to A_*(x_0, x_1) \to B_*(f_0(x_0), f_1(x_1)) \\ \mathcal{U}_*(A_0, A_1) &= A_0 \to A_1 \to \mathcal{U} \end{aligned}$$

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Proposition

Cofreely parametric categories with families with arrow types and a universe exist.

A problem with reflexivities and arrow types

To use internal parametricity, where any type comes with a reflexive relation, we try to add:

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\mathit{refl} &: (\Gamma : \mathit{Ob}) \to \mathit{Tm}((x : \Gamma), \Gamma_*[x, x]) \\
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We do not know how to define:

$$\begin{aligned} \mathit{refl}_{A \to B} &= ? \\ \mathit{refl}_{El(X)} &= ? \end{aligned}$$

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In Part 2 we consider models without arrow types or a universe.

Part 2:

Parametricity as a Module Structure

An alternative approach

Using extensions by section has drawbacks:

- Each example requires tedious work.
- Hard to prove that cubical models are cofreely parametric, because cofreely parametric models are complicated limits.

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We alleviate these using a symmetric monoidal closed category of models.

Plan for Part 2

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- Axiomatize parametric models as modules in a symmetric monoidal closed category.
- Give a convenient description for cofree modules.
- Prove that the following are cofree modules:
  - Categories of (many variants of) cubical objects.
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- Revisit parametric categories for inspiration.
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Remark

Use a strict variant of clans to get a symmetric monoidal closed structure.

Back to parametric categories

Definition

A parametric category is a category $\mathcal{C}$ equipped with:

- An endofunctor $\_{}_* : \mathcal{C} \to \mathcal{C}$.
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Functors from $\square$ to $\mathcal{C}$ are semi-cubical objects in $\mathcal{C}$.

Let $\mathcal{M}$ be a strict monoidal category.

Definition

An $\mathcal{M}$-module is a category $\mathcal{C}$ with a strict monoidal functor:

$$\alpha \quad : \quad \mathcal{M} \to \mathit{End}_{\mathcal{C}}$$

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In a $\square$-module $\mathcal{C}$, any X comes with:

$$\begin{aligned} F \quad &: \quad \square \to \mathcal{C} \\ F(i) \quad &= \quad \alpha(i)(X) \end{aligned}$$

giving a semi-cubical object with X as object of points.

Parametric models as modules

Let $\mathcal{V}$ be a symmetric monoidal closed category.

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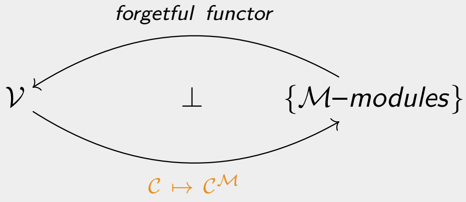
Example

$$\begin{aligned} \mathcal{V} &= \{\textit{Categories}\} \\ \mathcal{M} &= \square \\ \{\mathcal{M}\text{–}\textit{modules}\} &= \{\textit{Parametric categories}\} \end{aligned}$$

Cofreely parametric models

Theorem [folklore, e.g. Pareigis 77]



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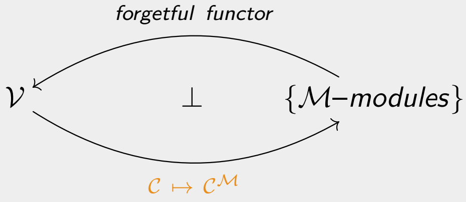

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We prove this for sets. But the proof is linear so it works in $\mathcal{V}$.

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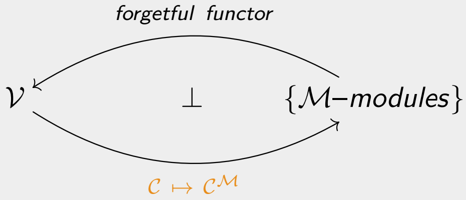

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Categories of semi-cubical objects are cofreely parametric.

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In a clan, types are represented by fibrations.

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Lemma

A notion of parametricity for strict clans consists of:

- A strict clan $\mathcal{M}$.
- A strict monoidal product on $\mathcal{M}$ commuting with limits in each variable.

Such that any $p : A \twoheadrightarrow \Gamma$ and $q : B \twoheadrightarrow \Delta$ induce a fibration:

$$p \odot q \quad : \quad A \otimes B \twoheadrightarrow A \otimes \Delta \underset{\Gamma \otimes \Delta}{\times} \Gamma \otimes B$$

Parametric clans and Reedy fibrant cubes

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Let $\square_c$ be the free monoidal strict clan generated by:

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Clans of Reedy fibrant semi-cubical objects are cofreely $\square_c$-parametric.

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Proof sketch

Fibrations in $\square_c$ are generated by the maps:

$$i \odot \cdots \odot i$$

which send a cube to its border.

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- Remove strictness assumptions by using a 2-category of models of type theory.

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Strategy

Axiomatize that Kan fibrations are stable by type constructors.

- Mix reflexivities with arrow types and a universe, inspired by:

Lemma

Let $\mathcal{C}$ be a category exponentials and enough limits,
then for any category $\square$, the category $\mathcal{C}^{\square}$ has exponentials.